

Finite size corrections in the random energy model and the replica approach

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We present a systematic and exact way of computing finite size corrections for the random energy model, in its low temperature phase. We obtain explicit (though complicated) expressions for the finite size corrections of the overlap functions. In its low temperature phase, the random energy model is known to exhibit Parisi's broken symmetry of replicas. The finite size corrections given by our exact calculation can be reproduced using replicas if we make specific assumptions about the fluctuations (with negative variances!) of the number and sizes of the blocks when replica symmetry is broken. As an alternative we show that the exact expression for the non-integer moments of the partition function can be written in terms of coupled contour integrals over what can be thought of as "complex replica numbers". Parisi's one step replica symmetry breaking arises naturally from the saddle point of these integrals without making any ansatz or using the replica method. The fluctuations of the "complex replica numbers" near the saddle point in the imaginary direction correspond to the negative variances we observed in the replica calculation. Finally our approach allows one to see why some apparently diverging series or integrals are harmless.

1. Introduction

Often the calculation of the extensive part of the free energy of mean field models can be reduced to finding the saddle point of some action which depends on an integer number (usually finite) of parameters (e.g. the energy or the magnetization). Then fluctuations can be calculated by replacing the action by its quadratic approximation near the saddle point. Expanding around the saddle point also enables the finite size corrections to be obtained. These are well known procedures which work well as long as the number of variables, on which the saddle point is calculated, is an integer. When one tries to apply the same ideas to the theory of disordered systems using the replica approach, the number of replicas is usually not an integer any more and the first difficulty one has to face is to give a meaning to a quadratic form with a

non-integer number of variables. The difficulty is even worse when the symmetry between this non-integer number of variables is broken as in Parisi's theory of mean field spin glasses.

In 1979-1980 Parisi [1, 2, 3] proposed a replica based solution of the Sherrington-Kirkpatrick [4, 5] mean field model of spin glasses. In Parisi's theory, the extensive part of the free energy could be determined by finding a saddle point in an unusual domain: it was a saddle point in the space of $n \times n$ matrices where the size n of the matrix was a continuous variable (in fact in the replica calculation one had to take the limit $n \rightarrow 0$ at the end of the calculation). Parisi was able to give a meaning to such a saddle point when n is not an integer.

Even before Parisi's work, for non-integer n , the Gaussian form around the saddle point was already understood in the replica symmetric phase by de Almeida and Thouless [6]. However, when replica symmetry is broken, determining the quadratic form around the saddle point is far from obvious [7]. This is why the form of the leading finite size corrections has been debated for a long time and has made it difficult to connect the theory with the results of numerical simulations [8, 9, 10, 11, 12, 13, 14, 15, 16]. Understanding the fluctuations near the saddle point is also a necessary step to build a field theory in finite dimension [17, 18].

In the present paper, we present a full analysis of the fluctuations near a saddle point with a broken replica symmetry for the random energy model, a spin glass model much simpler than the Sherrington Kirkpatrick model. Random energy models (REM) can be solved exactly [19, 20, 21, 22] without recourse to the replica method. But they can also be solved using replicas and they are among the simplest models for which Parisi's replica symmetry breaking [1, 2] scheme holds [20, 23]. However, computing finite size corrections using replicas has proved challenging even for simple models such as the REM [24, 25, 26]. We present here a systematic and direct way of computing the finite size corrections of random energy models in the broken replica symmetry phase. Although our calculations are done without recourse to replicas, our results can be interpreted as due to fluctuations of the parameters necessary to describe the broken replica symmetry.

Here we work with a Poisson version of the REM, which is exponentially close, for large system sizes, to the original REM (see Appendix A). How this Poisson REM is defined and how the overlaps or the moments of the partition function can be computed for this Poisson REM is the purpose of section 2. In section 3, we develop a systematic way of computing the finite size corrections of the overlaps and of the non-integer moments of the partition function. In section 4, we discuss how the results of section 3, for the finite size corrections of the overlaps, can be magenta reproduced using Parisi's broken replica symmetry magenta scheme. We show that to obtain the correct finite size corrections, one has to supplement Parisi's ansatz by fluctuations of the replica numbers, with negative variances. In section 5, we write exact expressions of the non-integer moments of the partition function as contour integrals over "complex replica numbers" (62). Then Parisi's ansatz appears naturally as the saddle point in these replica numbers, and the fluctuations calculated at this saddle point are consistent with the fluctuations predicted in the previous sections.

2. A Poisson process version of the random energy model

In this section we first recall a few known results on the random energy model. We then define a Poisson process version of the REM, for which we show how to compute the overlaps and the non-integer moments of the partition function. It is known that in the low temperature phase of the REM, one can represent the energies by a Poisson process [27, 28, 29]. While in the large N limit it is sufficient to take a Poisson process with an exponential density, here, because we are interested by finite size effects, we need to include

corrections to this exponential density.

2.1. The random energy model (REM)

In the random energy model, one considers a system with 2^N possible configurations \mathcal{C} , the energies $E(\mathcal{C})$ of which are i.i.d. random variables distributed according to a probability distribution

$$P(E(\mathcal{C})) = \frac{1}{\sqrt{\pi N J^2}} \exp \left\{ -\frac{E(\mathcal{C})^2}{N J^2} \right\} . \quad (1)$$

A sample is characterized by the choice of these 2^N random energies $E(\mathcal{C})$ and as usual in the theory of disordered systems the first quantity of interest is, for a typical sample, the free energy $F = \log Z(\beta)$ where

$$Z(\beta) = \sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})} \quad \text{with} \quad \beta = 1/T .$$

One of the remarkable features of the REM is that, in the large N limit, it undergoes a freezing transition at a critical temperature [19, 20]

$$T_c = J/(2\sqrt{\ln 2}) \quad (2)$$

and that, below this temperature, the partition function is dominated by the energies of the configurations close to the ground state [19, 20]

$$E_{\text{ground state}} \simeq -N J \sqrt{\ln 2} . \quad (3)$$

The REM is the simplest spin glass model which exhibits broken replica symmetry [20, 23]: the overlap $q(\mathcal{C}, \mathcal{C}')$ between two configurations can take only two possible values 0 or 1

$$q(\mathcal{C}, \mathcal{C}') = \delta_{\mathcal{C}, \mathcal{C}'}$$

and the Parisi's function $q(x)$ is a step function [23, 3]

$$q(x) = \Theta(x - 1 + \langle P_2 \rangle) \quad (4)$$

where $\Theta(x)$ is the Heaviside function, P_2 is the probability of finding, at equilibrium, two copies of the same sample in the same configuration

$$P_2 = \sum_{\mathcal{C}} \left(\frac{e^{-\beta E(\mathcal{C})}}{\sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})}} \right)^2 = \frac{Z(2\beta)}{Z(\beta)^2} \quad (5)$$

and $\langle . \rangle$ in (4) denotes an average over the samples, i.e. over the random energies $E(\mathcal{C})$.

In the large N limit, P_2 vanishes in the high temperature phase, while at low temperature (in the frozen phase) it takes non zero values with sample to sample fluctuations, because it is dominated by the ground state and the lowest excited states. In the $N \rightarrow \infty$ limit, direct calculations as well as replica calculations have shown [23, 30] that, below T_c ,

$$\langle P_2 \rangle = 1 - \mu \quad \text{with} \quad \mu = \frac{T}{T_c} = \frac{2\sqrt{\log 2}}{\beta J} \quad (6)$$

One of the goals of the present paper is to present a method to calculate the finite size corrections to this result in order to understand the effect of fluctuations in the space of replicas.

The quantity P_2 (which is nothing but the thermal average of the overlap $q(\mathcal{C}, \mathcal{C}')$) can be generalized to the probabilities P_k of finding k copies of the same sample in the same configuration

$$P_k = \sum_{\mathcal{C}} \left(\frac{e^{-\beta E(\mathcal{C})}}{\sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})}} \right)^k = \frac{Z(k\beta)}{Z(\beta)^k} \quad (7)$$

and generalized further to the probabilities P_{k_1, \dots, k_p} of finding k_1 copies in the same configuration, k_2 copies in a different configuration, \dots , k_p in yet another configuration

$$P_{k_1, \dots, k_p} = \frac{\sum_{\mathcal{C}_1, \dots, \mathcal{C}_p} e^{-\beta(k_1 E(\mathcal{C}_1) + \dots + k_p E(\mathcal{C}_p))}}{(\sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})})^{k_1 + \dots + k_p}} \quad (8)$$

where in the numerator of (8), the sum is over all possible sets of p different configurations $\mathcal{C}_1 \dots \mathcal{C}_p$. As for P_2 , the large N limits of the averages of these overlaps are known [31, 32, 3, 33]

$$\langle P_k \rangle = \frac{\Gamma(k - \mu)}{\Gamma(1 - \mu) \Gamma(k)} \quad ; \quad \langle P_{k_1, \dots, k_p} \rangle = \mu^{p-1} \frac{\Gamma(p)}{\Gamma(k_1 + \dots + k_p)} \frac{\Gamma(k_1 - \mu)}{\Gamma(1 - \mu)} \dots \frac{\Gamma(k_p - \mu)}{\Gamma(1 - \mu)} \quad (9)$$

and we will discuss below how to calculate their finite size corrections.

2.2. A Poisson process version of the REM

To slightly simplify the discussion below, we consider in the present paper a Poisson process version of the REM, *the Poisson REM*. The same idea of replacing a random energy model by a Poisson process was already used [34] for the DREM (a version of the REM where energies can take only integer values). In this Poisson REM, the values of the energies are the points generated by a Poisson process on the real line with intensity

$$\rho(E) = \frac{2^N}{\sqrt{\pi N J^2}} \exp \left\{ -\frac{E^2}{N J^2} \right\} . \quad (10)$$

This means that each infinitesimal interval $(E, E + \Delta E)$ on the real line is either empty, with probability $1 - \rho(E)\Delta E$, or occupied by a single configuration with probability $\rho(E)\Delta E$. As we eventually take $\Delta E \rightarrow 0$ it is justified to forget events where more than one level falls into the interval $(E, E + \Delta E)$.

One way of thinking of this Poisson REM is to divide the energy axis into intervals of size ΔE and label each interval with an integer $j \in (-\infty, +\infty)$. The energy associated with interval j is given by $j\Delta E$. A realisation of the disorder is given by a set of independent random binary variables $\{y_j\}$, which determine if the interval j contains an energy level

$$y_j = \begin{cases} 1 & \text{if interval } j \text{ contains an energy level,} \\ 0 & \text{if not .} \end{cases} \quad (11)$$

These independent random variables $\{y_j\}$ are chosen according to

$$y_j = \begin{cases} 1 & \text{with probability } \rho(j\Delta E) \Delta E \\ 0 & \text{with probability } 1 - \rho(j\Delta E) \Delta E . \end{cases} \quad (12)$$

The partition function, for a particular realisation $\{y_j\}$ of disorder, is given by

$$Z[\{y_j\}] = \sum_{j=-\infty}^{+\infty} y_j e^{-\beta j \Delta E}. \quad (13)$$

and the probability, at equilibrium, of finding the system in a specific energy interval is

$$\text{Pr}(\text{system in interval } i) = \frac{y_i e^{-\beta i \Delta E}}{\sum_{j=-\infty}^{+\infty} y_j e^{-\beta j \Delta E}}. \quad (14)$$

The Poisson REM has on average 2^N energy levels. In the large N limit, it has the same free energy as the REM (see appendix A), with in particular the same transition temperature (2). One difference, though, is that the total number of configurations fluctuates in the Poisson REM while it is fixed in the REM.

As for the REM, the low temperature phase of the Poisson REM is dominated by the energy levels close to the ground state and the average overlaps are given by (9) in the large N limit. In fact, as shown in appendix A, the difference between the free energies of REM and of the Poisson REM is exponentially small in the system size N . So we expect all the $1/N$ corrections to be the same for both models.

2.3. Expressions of the overlaps in the Poisson REM

In the Poisson REM, the probabilities P_k defined in (7) take the form (14)

$$P_k = \sum_{i=-\infty}^{+\infty} [\text{Pr}(\text{system in interval } i)]^k = \frac{1}{Z[\{y_j\}]^k} \sum_{i=-\infty}^{+\infty} y_i e^{-\beta k i \Delta E}. \quad (15)$$

One difficulty when one tries to average (15) over the $\{y_i\}$ is the presence of Z in the denominator. This difficulty can be overcome by using an integral representation of the Gamma function

$$\frac{1}{Z^k} = \frac{1}{\Gamma(k)} \int_0^\infty dt t^{k-1} e^{-Zt} = \frac{1}{\Gamma(k)} \int_0^\infty dt t^{k-1} \prod_{j=-\infty}^{\infty} \left[\exp\left(-t y_j e^{-\beta j \Delta E}\right) \right]. \quad (16)$$

Then using (12) to average (15) over the $\{y_i\}$ and taking the limit $\Delta E \rightarrow 0$ gives

$$\langle P_k \rangle = \frac{1}{\Gamma(k)} \int_0^\infty dt t^{k-1} F_k(t) \exp(-F(t)), \quad (17)$$

where

$$F(t) = \int_{-\infty}^{+\infty} \left(1 - \exp\left(-t e^{-\beta E}\right)\right) \rho(E) dE \quad (18)$$

and

$$F_k(t) = (-)^{k+1} \frac{d^k F(t)}{dt^k} = \int_{-\infty}^{+\infty} \exp\left(-k\beta E - t e^{-\beta E}\right) \rho(E) dE. \quad (19)$$

A similar calculation for the more general overlap (8) leads to

$$\langle P_{k_1, \dots, k_p} \rangle = \frac{1}{\Gamma(k_1 + \dots + k_p)} \int_0^\infty dt t^{k_1 + \dots + k_p - 1} F_{k_1}(t) \dots F_{k_p}(t) \exp(-F(t)). \quad (20)$$

Expressions (17-20) (and (22-25) below) are exact for a Poisson REM with an arbitrary $\rho(E)$.

2.4. The moments of Z and the weighted overlaps

As we will discuss below for the replica approach, it is also useful to obtain exact expressions for the moments of the partition function and for the weighted overlaps which will appear in the replica approach of section 4. Expressions of the integer or non-integer moments [35, 36] of Z are useful to calculate the fluctuations and the large deviations of the free energy [37]. To compute the non-integer moments of the partition function $\langle Z^m \rangle$ for $0 < m < 1$ we use again an integral representation similar to (16)

$$\langle Z^m \rangle = \frac{1}{\Gamma(-m)} \int_0^\infty dt t^{-m-1} \left(\langle e^{-tZ} \rangle - 1 \right). \quad (21)$$

(the calculation below could be extended to $m \notin (0, 1)$ by replacing (21) by the appropriate representation of the Gamma function.)

By averaging over the $\{y_i\}$ as above in (16,17) one gets an expression very similar to expression (3.8) of Ogure and Kabashima [34] for the DREM

$$\langle Z^m \rangle = \frac{1}{\Gamma(-m)} \int_0^\infty dt t^{-m-1} \left(\exp(-F(t)) - 1 \right). \quad (22)$$

One can also define generalized weighted overlaps (where events are weighted by powers of the partition function)

$$\langle P_k \rangle_m = \frac{\langle P_k Z^m \rangle}{\langle Z^m \rangle} \quad ; \quad \langle P_{k_1, \dots, k_p} \rangle_m = \frac{\langle P_{k_1, \dots, k_p} Z^m \rangle}{\langle Z^m \rangle}. \quad (23)$$

Using for $k \geq 1$ and $0 < m < 1$ the identity

$$Z^{m-k} = \frac{1}{\Gamma(k-m)} \int_0^\infty dt t^{k-m-1} e^{-tZ}$$

one gets

$$\langle P_k Z^m \rangle = \frac{1}{\Gamma(k-m)} \int_0^\infty dt t^{k-m-1} F_k(t) \exp(-F(t)), \quad (24)$$

and

$$\langle P_{k_1, \dots, k_p} Z^m \rangle = \frac{1}{\Gamma(k_1 + \dots + k_p - m)} \int_0^\infty dt t^{k_1 + \dots + k_p - m - 1} F_{k_1}(t) \dots F_{k_p}(t) \exp(-F(t)). \quad (25)$$

Formulas (22-25) are exact for the Poisson REM. They summarize all the previous ones in particular (17,20). For example one recovers (20) by taking the $m \rightarrow 0$ limit. They will be our starting point to calculate $1/N$ corrections.

3. Finite size corrections to the overlap functions

3.1. A direct calculation of finite size corrections

In the low temperature phase, the partition function of the REM or of the Poisson REM is dominated by the energies close to the ground state energy. It is therefore legitimate to replace the density $\rho(E)$ by an approximation valid in the neighbourhood of the ground state energy. Let us write (10) as

$$\rho(E) = A \exp [\alpha(E - E_0) - \epsilon(E - E_0)^2] \quad (26)$$

where we define (3)

$$E_0 = -NJ\sqrt{\log 2}, \quad (27)$$

$$\alpha = \frac{2|E_0|}{NJ^2} = \frac{2\sqrt{\log 2}}{J}, \quad (28)$$

$$\epsilon = \frac{1}{NJ^2}, \quad (29)$$

and

$$A = \frac{1}{\sqrt{\pi NJ^2}}. \quad (30)$$

In the REM, the distances between the energies of the ground state and of the lowest excited states remain of order 1 (in the large N limit) and $E_{\text{ground state}} - E_0 = \mathcal{O}(\log N)$ (see [20, 38, 25]) so that (26) is valid in the vicinity of the ground state.

Note that with these definitions (28), one has (6)

$$\mu = \frac{\alpha}{\beta}. \quad (31)$$

Therefore for ϵ small (i.e. for large N), one can replace (26) by

$$\rho(E) = A \exp [\alpha(E - E_0)] (1 - \epsilon(E - E_0)^2 + \mathcal{O}(\epsilon^2)) \quad (32)$$

and this can be written as

$$\rho(E) = A \left(1 - \epsilon \frac{d^2}{d\gamma^2} \right) e^{\gamma(E-E_0)} \Big|_{\gamma=\alpha} + \mathcal{O}(\epsilon^2), \quad (33)$$

so that (18,19) become

$$F(t) = \frac{A}{\alpha} \Gamma \left(1 - \frac{\alpha}{\beta} \right) t^{\frac{\alpha}{\beta}} e^{-\alpha E_0} - \epsilon \frac{d^2}{d\gamma^2} \left(\frac{A}{\gamma} \Gamma \left(1 - \frac{\gamma}{\beta} \right) t^{\frac{\gamma}{\beta}} e^{-\gamma E_0} \right) \Big|_{\gamma=\alpha} + \mathcal{O}(\epsilon^2), \quad (34)$$

$$F_k(t) = \frac{A}{\beta} \Gamma \left(k - \frac{\alpha}{\beta} \right) t^{\frac{\alpha}{\beta} - k} e^{-\alpha E_0} - \epsilon \frac{d^2}{d\gamma^2} \left(\frac{A}{\beta} \Gamma \left(k - \frac{\gamma}{\beta} \right) t^{\frac{\gamma}{\beta} - k} e^{-\gamma E_0} \right) \Big|_{\gamma=\alpha} + \mathcal{O}(\epsilon^2). \quad (35)$$

under the condition

$$\alpha < \beta \quad \text{i.e.} \quad \mu < 1. \quad (36)$$

Substituting the expansions (34,35) into the integral form (17) of $\langle P_k \rangle$ gives,

$$\langle P_k \rangle = \frac{\Gamma \left(k - \frac{\alpha}{\beta} \right)}{\Gamma(k) \Gamma \left(1 - \frac{\alpha}{\beta} \right)} + \frac{\epsilon A}{\alpha \Gamma(k) \Gamma \left(1 - \frac{\alpha}{\beta} \right)} \frac{d^2 B(\gamma)}{d\gamma^2} \Big|_{\gamma=\alpha} + \mathcal{O}(\epsilon^2) \quad (37)$$

where

$$B(\gamma) = \Gamma \left(\frac{\gamma}{\alpha} \right) \left[\frac{\alpha}{A \Gamma \left(1 - \frac{\alpha}{\beta} \right)} \right]^{\frac{\gamma}{\alpha}} \left[\Gamma \left(1 - \frac{\gamma}{\beta} \right) \Gamma \left(k - \frac{\alpha}{\beta} \right) - \Gamma \left(k - \frac{\gamma}{\beta} \right) \Gamma \left(1 - \frac{\alpha}{\beta} \right) \right].$$

This gives

$$\begin{aligned} \langle P_k \rangle = & \frac{\Gamma\left(k - \frac{\alpha}{\beta}\right)}{\Gamma(k) \Gamma\left(1 - \frac{\alpha}{\beta}\right)} + \epsilon \left[\left(2 \frac{\log(A\Gamma(1 - \frac{\alpha}{\beta})) - \Gamma'(1) - \log \alpha}{\alpha} + 2 \frac{\Gamma'(1 - \frac{\alpha}{\beta})}{\beta \Gamma(1 - \frac{\alpha}{\beta})} \right) \frac{d}{d\alpha} \left(\frac{\Gamma\left(k - \frac{\alpha}{\beta}\right)}{\Gamma(k) \Gamma\left(1 - \frac{\alpha}{\beta}\right)} \right) \right. \\ & \left. - \frac{d^2}{d\alpha^2} \left(\frac{\Gamma\left(k - \frac{\alpha}{\beta}\right)}{\Gamma(k) \Gamma\left(1 - \frac{\alpha}{\beta}\right)} \right) \right] + \mathcal{O}(\epsilon^2). \end{aligned} \quad (38)$$

We recover the known [31, 32, 30, 3, 33] zero-th order term (9). By replacing α by its expression (28) and by using the expression (31) for μ one gets the $1/N$ correction

$$\langle P_k \rangle = \frac{\Gamma(k - \mu)}{\Gamma(k) \Gamma(1 - \mu)} + \frac{1}{N} \left[\Delta_1 \frac{d}{d\mu} \left(\frac{\Gamma(k - \mu)}{\Gamma(k) \Gamma(1 - \mu)} \right) + \Delta_2 \frac{d^2}{d\mu^2} \left(\frac{\Gamma(k - \mu)}{\Gamma(k) \Gamma(1 - \mu)} \right) \right] + o\left(\frac{1}{N}\right) \quad (39)$$

where

$$\begin{aligned} \Delta_1 &= \frac{-\Gamma'(1) + \log(\Gamma(1 - \mu)) - \log(2\sqrt{N\pi \log 2})}{2 \log 2} \mu + \frac{\mu^2}{2 \log 2} \frac{\Gamma'(1 - \mu)}{\Gamma(1 - \mu)}, \\ \Delta_2 &= -\frac{1}{4 \log 2} \mu^2. \end{aligned}$$

The idea introduced in this section to compute $1/N$ corrections is straightforward enough to be extended to compute higher orders or the finite size corrections of other quantities, like the moments of the partition function (22) or the weighted generalized overlaps (23, 25).

3.2. An alternative way of computing finite size corrections

We discuss now an alternative way of computing the $1/N$ corrections which is somewhat simpler. One can rewrite (32) as

$$\rho(E) = \left\{ A e^{(\alpha + \phi)(E - E_0)} \right\}_{\phi} + \mathcal{O}(\epsilon^2) \quad (40)$$

where ϕ is a random variable (of order $\epsilon^{\frac{1}{2}}$) which satisfies

$$\{\phi\}_{\phi} = 0 \quad ; \quad \{\phi^2\}_{\phi} = -2\epsilon \quad (41)$$

and $\{.\}_{\phi}$ denotes an average over the variable ϕ . Negative variances appear here and in several other places in this paper. Here (41) simply means that for an arbitrary function $G(\phi)$ one has

$$\{G(\phi)\}_{\phi} = G(0) - \epsilon G''(0) + \mathcal{O}(\epsilon^2) \quad (42)$$

(alternatively one could think of ϕ as being a pure imaginary random number).

Using (40,41,42) in (18,19) one gets for $F(t)$ and the $F_{k_i}(t)$

$$\begin{aligned} F(t) &= -\frac{A}{\beta} \left\{ \Gamma(-\mu_0) t^{\mu_0} e^{-\beta \mu_0 E_0} \right\}_{\phi_0} + \mathcal{O}(\epsilon^2) \\ F_{k_i}(t) &= \frac{A}{\beta} \left\{ \Gamma(k_i - \mu_i) t^{\mu_i - k_i} e^{-\beta \mu_i E_0} \right\}_{\phi_i} + \mathcal{O}(\epsilon^2) \end{aligned}$$

where for $1 \leq i \leq p$

$$\mu_i = \frac{\alpha + \phi_i}{\beta} = \mu + \frac{\phi_i}{\beta} .$$

As at order ϵ one has

$$e^{-F(t)} = e^{-F^*(t)}(1 - F(t) + F^*(t)) + \mathcal{O}(\epsilon^2)$$

where

$$F^*(t) = -\frac{A}{\beta} \Gamma(-\mu) t^\mu e^{-\beta \mu E_0}$$

and this gives for the weighted overlaps (20) using the fact that the difference $F - F^*$ is of order ϵ

$$\begin{aligned} \langle P_{k_1, \dots, k_p} Z^m \rangle &= \left(\frac{A}{\beta} \right)^p e^{-\beta m E_0} \left[\left\{ \frac{\Gamma(k_1 - \mu_1) \cdots \Gamma(k_p - \mu_p) \Gamma(\frac{\mu_1 + \dots + \mu_p - m}{\mu})}{\mu \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{\frac{m - \mu_1 - \dots - \mu_p}{\mu}} \right\}_{\phi_1, \dots, \phi_p} \right. \\ &+ \left(\frac{A}{\beta} \right) \frac{\Gamma(k_1 - \mu) \cdots \Gamma(k_p - \mu) \Gamma(p + 1 - \frac{m}{\mu})}{\mu \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{\frac{m}{\mu} - p - 1} (-\Gamma(-\mu)) \\ &- \left(\frac{A}{\beta} \right) \left\{ \frac{\Gamma(k_1 - \mu) \cdots \Gamma(k_p - \mu) \Gamma(p + \frac{\mu_0 - m}{\mu})}{\mu \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{\frac{m}{\mu} - p - \frac{\mu_0}{\mu}} (-\Gamma(-\mu_0)) \right\}_{\phi_0} \Bigg] \\ &+ \mathcal{O}(\epsilon^2) . \end{aligned} \quad (43)$$

The expression for $\langle Z^m \rangle$ turns out to be a special case ($p = 0$) of (43) and therefore at order ϵ

$$\begin{aligned} \langle P_{k_1, \dots, k_p} \rangle_m &= \left(\frac{A}{\beta} \right)^p \Gamma(-m) \left[\left\{ \frac{\Gamma(k_1 - \mu_1) \cdots \Gamma(k_p - \mu_p) \Gamma(\frac{\mu_1 + \dots + \mu_p - m}{\mu})}{\Gamma(-\frac{m}{\mu}) \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{-\frac{\mu_1 + \dots + \mu_p}{\mu}} \right\}_{\phi_1, \dots, \phi_p} \right. \\ &+ \left(\frac{A}{\beta} \right) \frac{\Gamma(k_1 - \mu) \cdots \Gamma(k_p - \mu) \Gamma(p + 1 - \frac{m}{\mu})}{\Gamma(-\frac{m}{\mu}) \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{-p - 1} (-\Gamma(-\mu)) \\ &- \left(\frac{A}{\beta} \right) \left\{ \frac{\Gamma(k_1 - \mu) \cdots \Gamma(k_p - \mu) \Gamma(p + \frac{\mu_0 - m}{\mu})}{\Gamma(-\frac{m}{\mu}) \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{-p - \frac{\mu_0}{\mu}} (-\Gamma(-\mu_0)) \right\}_{\phi_0} \\ &+ \left(\frac{A}{\beta} \right) \left\{ \frac{\Gamma(k_1 - \mu) \cdots \Gamma(k_p - \mu) \Gamma(p - \frac{m}{\mu}) \Gamma(\frac{\mu_0 - m}{\mu})}{\Gamma(-\frac{m}{\mu})^2 \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{-p - \frac{\mu_0}{\mu}} (-\Gamma(-\mu_0)) \right\}_{\phi_0} \\ &- \left(\frac{A}{\beta} \right) \frac{\Gamma(k_1 - \mu) \cdots \Gamma(k_p - \mu) \Gamma(p - \frac{m}{\mu}) \Gamma(1 - \frac{m}{\mu})}{\Gamma(-\frac{m}{\mu})^2 \Gamma(k_1 + \dots + k_p - m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{-p - 1} (-\Gamma(-\mu)) \Bigg] \\ &+ \mathcal{O}(\epsilon^2) . \end{aligned}$$

After a (tedious but) straightforward calculation where we have used two simple properties ($\Gamma'(z + 1) =$

$z\Gamma'(z) + \Gamma(z)$ and $\Gamma''(z+1) = z\Gamma''(z) + 2\Gamma'(z)$ of Gamma functions one gets

$$\begin{aligned} \langle P_{k_1, k_2, \dots, k_p} \rangle_m &= (-)^p \frac{\Gamma(p - \frac{m}{\mu})}{\Gamma(-\frac{m}{\mu})} \frac{\Gamma(-m)}{\Gamma(k_1 + \dots + k_p - m)} \frac{\Gamma(k_1 - \mu)}{\Gamma(-\mu)} \dots \frac{\Gamma(k_p - \mu)}{\Gamma(-\mu)} \\ &\times \left[1 + 2 \frac{\epsilon}{\beta^2} \log \left(-\frac{A}{\beta} \Gamma(-\mu) \right) \left(-\frac{\Sigma_1}{\mu} + \frac{p}{\mu} \frac{\Gamma'(-\mu)}{\Gamma(-\mu)} - \frac{m}{\mu^3} \frac{\Gamma'(-\frac{m}{\mu})}{\Gamma(-\frac{m}{\mu})} + \frac{m}{\mu^3} \frac{\Gamma'(p - \frac{m}{\mu})}{\Gamma(p - \frac{m}{\mu})} \right) \right. \\ &\quad + \frac{\epsilon}{\beta^2} \left(-\Sigma_2 + 2 \frac{\Sigma_1}{\mu} \frac{\Gamma'(p - \frac{m}{\mu})}{\Gamma(p - \frac{m}{\mu})} - \frac{2m\Gamma'(-\mu)\Gamma'(-\frac{m}{\mu})}{\mu^2\Gamma(-\mu)\Gamma(-\frac{m}{\mu})} + \frac{2m\Gamma'(-\mu)\Gamma'(p - \frac{m}{\mu})}{\mu^2\Gamma(-\mu)\Gamma(p - \frac{m}{\mu})} \right. \\ &\quad \left. \left. - \frac{2p\Gamma'(-\mu)\Gamma'(p - \frac{m}{\mu})}{\mu\Gamma(-\mu)\Gamma(p - \frac{m}{\mu})} + p \frac{\Gamma''(-\mu)}{\Gamma(-\mu)} + \frac{m\Gamma''(-\frac{m}{\mu})}{\mu^3\Gamma(-\frac{m}{\mu})} - \frac{2\Gamma'(-\frac{m}{\mu})}{\mu^2\Gamma(-\frac{m}{\mu})} - \frac{m\Gamma''(p - \frac{m}{\mu})}{\mu^3\Gamma(p - \frac{m}{\mu})} + \frac{2\Gamma'(p - \frac{m}{\mu})}{\mu^2\Gamma(p - \frac{m}{\mu})} \right) \right] \\ &\quad + \mathcal{O}(\epsilon^2) \end{aligned} \quad (44)$$

where

$$\Sigma_1 = \sum_{i=1}^p \frac{\Gamma'(k_i - \mu)}{\Gamma(k_i - \mu)} \quad ; \quad \Sigma_2 = \sum_{i=1}^p \frac{\Gamma''(k_i - \mu)}{\Gamma(k_i - \mu)}$$

We checked that for $p = 1$ this formula reduces to (38) in the limit $m \rightarrow 0$.

Using expressions (6,29) and (30) for μ, ϵ and A gives the $1/N$ corrections in terms of T and T_c .

3.3. The non-integer moments $\langle Z^m \rangle$ of the partition function

A by-product of the above calculation (obtained by setting $p = 0$ in (43)), is the expression of the non-integer moments $\langle Z^m \rangle$ for $0 < m < 1$. At leading order in ϵ it gives

$$\langle Z^m \rangle = e^{-\beta m E_0} \frac{\Gamma(-\frac{m}{\mu})}{\mu \Gamma(-m)} \left(-\frac{A}{\beta} \Gamma(-\mu) \right)^{\frac{m}{\mu}} + \mathcal{O}(\epsilon) . \quad (45)$$

Then replacing A by its expression (30)

$$\langle Z^m \rangle_{\text{exact}} \simeq \frac{1}{\mu} \frac{\Gamma(-\frac{m}{\mu})}{\Gamma(-m)} (N\pi\beta^2 J^2)^{-\frac{m}{2\mu}} (-\Gamma(-\mu))^{\frac{m}{\mu}} e^{N\beta m J \sqrt{\log 2}} + \mathcal{O}(\epsilon) . \quad (46)$$

This expression is obtained under the condition (36) for $0 < m < \mu = T/T_c < 1$. In the limit $m \rightarrow 0$ one recovers the free energy [19, 20, 38, 39]. For $m > 0$, the N dependence is also the same as in [36]. If the condition $0 < m < \mu = T/T_c < 1$ is not satisfied then one would need to expand $\rho(E)$ around an energy different from E_0 (see (32)). For example, for $\mu > 1$, that is in the high temperature phase, the configurations which contribute most are those with an energy $\simeq -NJ^2/2/T$ (see [20]) and one could in principle repeat the above calculation (done for $\mu = T/T_c < 1$) by starting with the approximation (26) with $E_0 = -NJ^2/2/T$.

4. The replica approach for the overlaps

In this section we are going to see that expression (44) obtained by a direct calculation without use of replicas is fully consistent with a broken symmetry of replicas when one lets the number of blocks and the sizes of the blocks fluctuate (with negative variances).

4.1. The Parisi ansatz

In the Parisi replica approach [1, 2, 3] to compute $\langle Z^m \rangle$, the symmetry between the m replicas is broken, meaning that the m replicas are grouped into blocks. For example, at the level of a single step of symmetry breaking, (for the REM it is well known that a single step is sufficient [23, 40, 41]) this means that $\langle Z^m \rangle$ is dominated by situations where the m replicas are grouped into r blocks of μ replicas. Then the weighted overlaps are given by

$$\langle P_{k_1, \dots, k_p} \rangle_m = \left(\frac{r!}{(r-p)!} \right) \left(\prod_{i=1}^p \frac{\mu!}{(\mu - k_i)!} \right) \left(\frac{(m - k_1 - \dots - k_p)!}{m!} \right). \quad (47)$$

Expression (47) as well as its generalization (48) will be established in section 4.2. In short, the first factor in (47) counts the number of ways of choosing p blocks among the r blocks, the product counts the number of ways of choosing k_1, \dots, k_p replica in each of the p blocks of μ replica each, the last term is the normalization which corresponds to the number of ways of choosing $k_1 + \dots + k_p$ replicas among m .

When p, k_1, k_2, \dots, k_p are integers, (47) is a rational function of the parameters m, r and μ . Therefore it can be analytically continued to non-integer values of these parameters m, r and μ and coincides with the following rational function

$$\langle P_{k_1, \dots, k_p} \rangle_m = \left(\frac{\Gamma(p-r)}{-\Gamma(-r)} \right) \left(\prod_{i=1}^p \frac{\Gamma(k_i - \mu)}{-\Gamma(-\mu)} \right) \left(\frac{-\Gamma(-m)}{\Gamma(k_1 + \dots + k_p - m)} \right) \quad (48)$$

If all blocks have the same size μ , the number of blocks is obviously

$$r = \frac{m}{\mu}$$

and with this choice, one can see that (48) reduces to the leading order of (44). So the broken replica symmetry does give the correct expression for the large N limit of the overlaps.

4.2. Letting the number of blocks and their sizes fluctuate

Now we want to let the number r of blocks, and the numbers μ_1, \dots, μ_k of replicas in these blocks fluctuate. Then (48) becomes

$$\langle P_{k_1, \dots, k_p} \rangle_m = \left\langle \left(\frac{\Gamma(p-r)}{-\Gamma(-r)} \right) \left(\prod_{i=1}^p \frac{\Gamma(k_i - \mu_i)}{-\Gamma(-\mu_i)} \right) \left(\frac{-\Gamma(-m)}{\Gamma(k_1 + \dots + k_p - m)} \right) \right\rangle_{r, \mu_1, \dots, \mu_p}. \quad (49)$$

Derivation of (49): Let us now explain how (49) can be derived. For the Poisson REM one can write the following exact expression of $\langle Z^m \rangle$ when m is an integer,

$$\langle Z^m \rangle = \sum_{r \geq 1} \frac{m!}{r!} \sum_{\mu_1 \geq 1} \dots \sum_{\mu_r \geq 1} \frac{\Psi(\mu_1)}{\mu_1!} \dots \frac{\Psi(\mu_r)}{\mu_r!} \delta[m, \mu_1 + \dots + \mu_r] \quad (50)$$

where

$$\Psi(\mu) = \int \rho(E) e^{-\mu \beta E} dE$$

One can evaluate in the same way, still for integer m ,

$$\langle Z^m P_{k_1, \dots, k_p} \rangle = \sum_{r \geq 1} \frac{(m - k_1 - \dots - k_p)!}{(r - p)!} \sum_{\mu_1 \geq 1} \dots \sum_{\mu_r \geq 1} \frac{\Psi(\mu_1)}{(\mu_1 - k_1)!} \dots \frac{\Psi(\mu_p)}{(\mu_p - k_p)!} \frac{\Psi(\mu_{p+1})}{\mu_{p+1}!} \dots \frac{\Psi(\mu_r)}{\mu_r!} \delta[m, \mu_1 + \dots + \mu_r] \quad (51)$$

Here the convention is $(-n)! = \infty$ for $n = 1, 2, \dots$. Taking the ratio of (51) and (50) one gets

$$\langle P_{k_1, \dots, k_p} \rangle_m \equiv \frac{\langle Z^m P_{k_1, \dots, k_p} \rangle}{\langle Z^m \rangle} = \left\langle \frac{(m - k_1 - \dots - k_p)!}{m!} \left(\prod_{i=1}^p \frac{\mu_i!}{(\mu_i - k_i)!} \right) \frac{r!}{(r - p)!} \right\rangle_{r, \mu_1, \dots, \mu_p} \quad (52)$$

where $\langle \cdot \rangle_{r, \mu_1, \dots, \mu_p}$ simply means an average over r, μ_1, \dots, μ_p . That is, for a test function G ,

$$\langle G(r, \mu_1, \dots, \mu_p) \rangle_{r, \mu_1, \dots, \mu_p} = \frac{1}{\langle Z^m \rangle} \sum_{r \geq 1} \frac{m!}{r!} \sum_{\mu_1 \geq 1} \dots \sum_{\mu_r \geq 1} \frac{\Psi(\mu_1)}{\mu_1!} \dots \frac{\Psi(\mu_r)}{\mu_r!} \delta[m, \mu_1 + \dots + \mu_r] G(r, \mu_1, \dots, \mu_p) .$$

Expression (52) is exact for any positive integer m . It has been derived when all the parameters are integers. For fixed integer values of p, k_1, \dots, k_p , it is a rational function of the parameters m, μ_1, \dots, μ_p . Therefore it can be analytically continued to non integer values of these parameters and it coincides with (49). This completes our derivation of (47,48,49).

1

4.3. Characteristics of the fluctuations

We now try to see in (49) what kind of fluctuations of the number r of blocks and the numbers $\mu_1 \dots \mu_k$ of replicas in each block would enable us to recover the finite size corrections obtained in (44) by a direct calculation.

We have checked, by "a tedious but straightforward calculation" that if we write for $1 \leq i \leq k$

$$\begin{aligned} \mu_i &= \mu + \psi_i \\ r &= \frac{m}{\mu} + \rho \end{aligned} \quad (53)$$

(49) becomes equivalent to (44) provided that

$$\begin{aligned} \langle \psi_i \rangle &= \epsilon \left[\frac{2}{\beta^2 \mu} \log \left(-\frac{A}{\beta} \Gamma(-\mu) \right) + \frac{2}{\beta^2} \frac{\Gamma'(-\mu)}{\Gamma(-\mu)} - \frac{2}{\beta^2 \mu} \frac{\Gamma'(-\frac{m}{\mu})}{\Gamma(-\frac{m}{\mu})} \right] \\ \langle \rho \rangle &= -\frac{m}{\mu^2} \langle \psi_i \rangle - \frac{2}{\mu^2 \beta^2} \epsilon \\ \langle \psi_i^2 \rangle &= -\frac{2}{\beta^2} \epsilon \\ \langle \psi_i \psi_j \rangle &= 0 \\ \langle \rho \psi_i \rangle &= \frac{2}{\mu \beta^2} \epsilon \\ \langle \rho^2 \rangle &= -\frac{2m}{\mu^3 \beta^2} \epsilon . \end{aligned} \quad (54)$$

In terms of $\mu = T/T_c$, $\beta = 1/T$ and N these expressions become

$$\begin{aligned}
\langle \psi_i \rangle &= \frac{2}{N\beta^2 J^2 \mu} \left[\log \left(-\frac{\Gamma(-\mu)}{\beta J \sqrt{N\pi}} \right) + \frac{\mu \Gamma'(-\mu)}{\Gamma(-\mu)} - \frac{\Gamma'(-\frac{m}{\mu})}{\Gamma(-\frac{m}{\mu})} \right] \\
\langle \rho \rangle &= -\frac{m}{\mu^2} \langle \psi_i \rangle - \frac{2}{N\mu^2 \beta^2 J^2} \\
\langle \psi_i^2 \rangle &= -\frac{2}{N\beta^2 J^2} \\
\langle \psi_i \psi_j \rangle &= 0 \\
\langle \rho \psi_i \rangle &= \frac{2}{N\mu \beta^2 J^2} \\
\langle \rho^2 \rangle &= -\frac{2m}{N\mu^3 \beta^2 J^2} .
\end{aligned} \tag{55}$$

So the $1/N$ corrections we calculated directly in (44) can indeed be interpreted as fluctuations of the number r of blocks and of the sizes μ_i of the blocks in Parisi's ansatz. The only price we pay is to allow negative variances.

5. The exact non-integer moments of the partition function written in terms of "complex replica numbers"

In this section we write the exact expression (46) of $\langle Z^m \rangle$ for the non-integer moments of the partition function (which was obtained without using replicas) in terms of contour integrals over what can be thought of as "complex replica numbers". We then show that one step replica symmetry breaking arises naturally from the saddle point of these integrals without making the Parisi ansatz. We also find that, without any additional assumptions, the fluctuations of the "complex replica numbers" in the imaginary direction correspond to the negative variances (55) we observed in the replica calculation of section 4 .

5.1. The contour integral representation of the non-integer moments $\langle Z^m \rangle$ for $0 < m < 1$

Our starting point is the following representation, valid for $0 < m < 1$, of the non integer moments

$$\langle Z^m \rangle = \frac{1}{\Gamma(-m)} \int_0^\infty dt \, t^{-m-1} (\langle e^{-tZ} \rangle - 1) . \tag{56}$$

We have seen (21,22) that

$$\langle e^{-tZ} \rangle = \exp \left[\int \rho(E) dE \left(\exp[-te^{-\beta E}] - 1 \right) \right] . \tag{57}$$

We now need to use the identity

$$\sum_{p \geq k} f(p) \frac{(-1)^p}{p!} = - \int_{\mathcal{C}_k} \frac{dz}{2\pi i} \Gamma(-z) f(z) \tag{58}$$

where the contour \mathcal{C}_k starts at $+\infty + i0$ and ends at $+\infty - i0$ and crosses the real axis between $k-1$ and k (see figure 1).

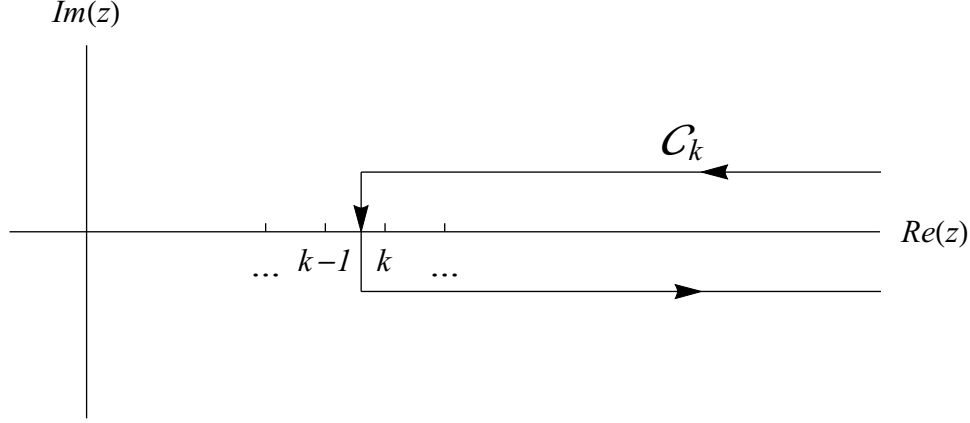


Figure 1: The contour \mathcal{C}_k starts at $+\infty + i0$ and ends at $+\infty - i0$ and crosses the real axis between $k-1$ and k .

This identity is valid for any analytic function $f(z)$ such that the sum and the integral in (58) converge. Using (58) in (56,57) (see the discussion on convergence in section 5.4 below) one can see that

$$\langle Z^m \rangle = \frac{-1}{\Gamma(-m)} \int_0^\infty dt \, t^{-m-1} \int_{\mathcal{C}_1} \frac{dr}{2\pi i} \Gamma(-r) \left(- \int \rho(E) dE \left(\exp[-te^{-\beta E}] - 1 \right) \right)^r$$

Using again the identity (58) one gets

$$\langle Z^m \rangle = -\frac{1}{\Gamma(-m)} \int_0^\infty dt \, t^{-m-1} \int_{\mathcal{C}_1} \frac{dr}{2\pi i} \Gamma(-r) \left(\int_{\mathcal{C}_1} \frac{d\mu}{2\pi i} \Gamma(-\mu) t^\mu \int dE \rho(E) e^{-\beta \mu E} \right)^r. \quad (59)$$

Let us assume that

$$\int \rho(E) dE \, e^{-\beta \mu E} = B \exp[N\phi(\mu)]. \quad (60)$$

For example, for the Poisson REM this gives (10)

$$B = 1 \quad ; \quad \phi(\mu) = \log 2 + \frac{\beta^2 J^2 \mu^2}{4}. \quad (61)$$

Making the change of variables $t = e^{Nx}$ equation (59) becomes

$$\langle Z^m \rangle = -\frac{N}{\Gamma(-m)} \int_{-\infty}^\infty dx \, e^{-Nxm} \int_{\mathcal{C}_1} \frac{dr}{2\pi i} \Gamma(-r) \left(B \int_{\mathcal{C}_1} \frac{d\mu}{2\pi i} \Gamma(-\mu) e^{N(x\mu + \phi(\mu))} \right)^r. \quad (62)$$

This expression is exact. Our goal now is to get its large N behaviour. To do so we found it more convenient to perform the saddle point calculation in the following order: first the integral over μ , then the integral over x , then the integral over r .

For large N , we evaluate the integral over μ using a saddle point, at a value of $0 < \mu < 1$ on the real axis to give

$$\langle Z^m \rangle = -\frac{N}{\Gamma(-m)} \int_{-\infty}^\infty dx \, e^{-Nmx} \int_{\mathcal{C}_1} \frac{dr}{2\pi i} \Gamma(-r) \left(\frac{-\Gamma(-\mu) B}{\sqrt{2\pi N \phi''(\mu)}} \right)^r e^{N(x\mu + \phi(\mu))r} \quad (63)$$

where the saddle point value μ has become a function of x and is solution of

$$x + \phi'(\mu) = 0 . \quad (64)$$

Now that μ is a function of x , one can calculate in (63) the saddle point with respect to x which is determined (together with (64)) by

$$r\mu - m = 0 . \quad (65)$$

Equations (64) and (65) give the saddle point values μ and x in terms of r

$$\mu = \frac{m}{r} \quad ; \quad x = -\phi' \left(\frac{m}{r} \right) \quad (66)$$

so that (63) becomes

$$\langle Z^m \rangle \simeq \frac{-1}{\Gamma(-m)} \int_{\mathcal{C}_1} \frac{dr}{i\sqrt{2\pi}} \Gamma(-r) \frac{\sqrt{N\phi''(\mu)}}{\sqrt{r}} \left(\frac{-\Gamma(-\mu) B}{\sqrt{2\pi N\phi''(\mu)}} \right)^r e^{Nr\phi(\frac{m}{r})} \quad (67)$$

where we have used (see (64)) that $1 + \phi''(\mu) \frac{d\mu}{dx} = 0$.

Now looking for the saddle point in r one gets that it should satisfy

$$\phi \left(\frac{m}{r} \right) - \frac{m}{r} \phi' \left(\frac{m}{r} \right) = 0 \quad \text{i.e.} \quad \phi(\mu) - \mu\phi'(\mu) = 0 \quad (68)$$

and (67) becomes

$$\langle Z^m \rangle \simeq \frac{r}{m} \frac{\Gamma(-r)}{\Gamma(-m)} \left(\frac{-\Gamma(-\mu) B}{\sqrt{2\pi N\phi''(\mu)}} \right)^r e^{Nr\phi(\frac{m}{r})} = \frac{1}{\mu} \frac{\Gamma(-\frac{m}{\mu})}{\Gamma(-m)} B^{\frac{m}{\mu}} \left(\frac{-\Gamma(-\mu)}{\sqrt{2\pi N\phi''(\mu)}} \right)^{\frac{m}{\mu}} e^{N\frac{m}{\mu}\phi(\mu)} . \quad (69)$$

This is our result, when $0 < m < \mu < 1$, for the non integer moments $\langle Z^m \rangle$ with ϕ given by (60) or (61) and μ the solution of (68). It was obtained without using the Parisi's symmetry breaking scheme. However, as we discuss further below, the correspondence between the m , r and μ integration variables used here and the replica numbers used in section 4 (see for example formula 50) is rather compelling. For this reason we will sometimes refer to the m , r and μ used in this section as "complex replica numbers".

5.2. Some remarks on the replica calculation

At this point we would like to make some remarks on the significance of the results from this approach to the replica calculation:

Remark 1: The above saddle point estimate is legitimate only if the saddle point value of r is between 0 and 1 (i.e. in the range where the contour \mathcal{C}_1 crosses the real axis). If not one can deform the contour to pass through the saddle point but one should not forget the contribution of the poles at the integer values of r . This is what happens in particular in the high temperature phase [24, 42].

Remark 2 : A similar calculation could be done for $m > 1$. The difference would be to replace the integration contour \mathcal{C}_1 of r in (59) by \mathcal{C}_k for $k - 1 < m < k$. The rest of the calculation would be very similar.

Remark 3 : A formula similar to (58) was already used in [25] (see also [34]). Here we use it twice starting from an exact expression and the one step replica symmetry breaking structure arises naturally as the saddle point of an exact expression (62) in the large N limit. The finite size corrections follow from fluctuations in the "complex replica numbers" about this saddle point.

Remark 4 : Several works in the past [43, 44, 45] have discussed the necessity of calculating the fluctuations around the saddle point to obtain the leading finite size corrections, using in particular linear response theory to determine the fluctuations of the Parisi function. The main difference between these works and our present approach is that our starting point to calculate finite size corrections is not based on Parisi's ansatz. It would be nice to see whether our results (55) could be recovered as the large p limit in the replica calculations [43, 44] of the p spin models.

Remark 5 : For the REM (61) the saddle point equations (68) gives for $\mu = 2\sqrt{\log 2}/\beta J$ which agrees with the definition of μ in (6). Then as $B = 1$, one can check that (69) does coincide with (46). So the exact "complex replica" approach based on (62) indeed agrees with the direct calculation leading to (46). Note that to get the right prefactor in (62) it was necessary to integrate over the fluctuations of the parameters r, μ and x meaning that we had to include the fluctuations around Parisi's ansatz.

5.3. How the replica numbers became complex

There is a remarkable similarity between the expressions (which are both exact) (50) and (59) of $\langle Z^m \rangle$ for integer and non-integer m : in (59) the number r of blocks and the sizes μ_1, \dots, μ_r of the blocks are not integer anymore (they have even become complex!). Going from integer m to non-integer m , one has to replace the measure (50)

$$\sum_{r \geq 1} \frac{m!}{r!} \sum_{\mu_1 \geq 1} \dots \sum_{\mu_r \geq 1} \frac{\delta[m, \mu_1 + \dots + \mu_r]}{\mu_1! \dots \mu_r!}$$

by (59)

$$\frac{-1}{2\pi i \Gamma(-m)} \int_0^\infty dt \, t^{-m-1} \int_{\mathcal{C}_1} dr \, \Gamma(-r) \left(\frac{1}{2\pi i} \right)^r \int_{\mathcal{C}_1} d\mu_1 \, \Gamma(-\mu_1) \dots \int_{\mathcal{C}_1} d\mu_r \, \Gamma(-\mu_r) \, t^{\mu_1 + \dots + \mu_r} .$$

These two measures can be rewritten respectively as

$$\int_{-\infty}^\infty \frac{dx}{2\pi} e^{-mix} \sum_{r \geq 1} \frac{m!}{r!} \sum_{\mu_1 \geq 1} \dots \sum_{\mu_r \geq 1} \frac{e^{ix(\mu_1 + \dots + \mu_r)}}{\mu_1! \dots \mu_r!} \quad (70)$$

and

$$- \int_{-\infty}^\infty \frac{dx}{2\pi i} e^{-mx} \int_{\mathcal{C}_1} dr \, \frac{\Gamma(-r)}{\Gamma(-m)} \int_{\mathcal{C}_1} d\mu_1 \, \frac{\Gamma(-\mu_1)}{2\pi i} \dots \int_{\mathcal{C}_1} d\mu_r \, \frac{\Gamma(-\mu_r)}{2\pi i} e^{x(\mu_1 + \dots + \mu_r)} . \quad (71)$$

Comparing these two expressions, we see that, up to factors i or -1 , the sums have become integrals, the integers r, μ_1, μ_p have become complex, and the inverse factorials $1/n!$ have been replaced by $\frac{\Gamma(-n)}{2\pi i}$. For the REM, this may shed some light on the mystery of Parisi's theory.

5.4. Why the diverging integrals or series are harmless

In the last step to obtain (59) we wrote

$$\int \rho(E) dE \left(\exp[-te^{-\beta E}] - 1 \right) = \int_{\mathcal{C}_1} \frac{d\mu}{2\pi i} \Gamma(-\mu) t^\mu \int dE \rho(E) e^{-\beta \mu E} \quad (72)$$

which in the case of the REM (61) gives

$$\int \rho(E) dE \left(\exp[-te^{-\beta E}] - 1 \right) = \int_{\mathcal{C}_1} \frac{d\mu}{2\pi i} \Gamma(-\mu) t^\mu 2^N \exp \left[\mu^2 \frac{N\beta^2 J^2}{4} \right]. \quad (73)$$

The integral in the r.h.s. of (73) clearly diverges as $\mu \rightarrow \infty \pm i0$.

The same would be true, in the limit $\mu \rightarrow \infty$, for the following power series expansion

$$\int \rho(E) dE \left(\exp[-te^{-\beta E}] - 1 \right) = \sum_{\mu \geq 1} \frac{(-t)^\mu}{\mu!} 2^N \exp \left[\mu^2 \frac{N\beta^2 J^2}{4} \right] \quad (74)$$

We think that these divergences are harmless for the following reason: we know that in the low temperature phase of the REM, everything is dominated by the energies close to E_0 given by (27). Therefore the $1/N$ corrections of the present paper would remain unchanged if one would replace the density $\rho(E)$ by a density $\tilde{\rho}(E)$ which is identical to $\rho(E)$ in the neighborhood of E_0 . For example we could choose

$$\tilde{\rho}(E) = \begin{cases} \rho(E) & \text{if } |E| < 2|E_0| \\ 0 & |E| > 2|E_0| \end{cases}.$$

With the distribution $\tilde{\rho}(E)$ the above integral and sum would become convergent while none of our results would be modified.

Diverging series appear frequently in replica calculations [46, 47, 48] in particular in the context of the KPZ equation and it would of course be interesting to see whether a similar reason could be invoked there to justify the manipulation of diverging series or integrals.

5.5. The fluctuations close to the saddle point

In the derivation of (69), we started from the exact expression (62) and we performed three saddle point calculations. The saddle point values were given by (64,65,68)

$$x = -\phi'(\mu) \quad ; \quad r = \frac{m}{\mu} \quad ; \quad \phi(\mu) - \mu\phi'(\mu) = 0.$$

One can now try to characterize the fluctuations responsible of the large N corrections at these saddle points.

If we apply the formulas derived in Appendix B to the integral (67) over r , one gets for the fluctuations (53) near this saddle point, after replacing r by m/μ

$$\langle \rho \rangle = \frac{1}{N} \left[-\frac{r^2}{m^2 \phi''(\mu)} - \frac{r^2}{2m} \frac{\phi'''(\mu)}{\phi''(\mu)^2} - \frac{r^3}{m^2 \phi''(\mu)} \left(-\frac{\Gamma'(-\frac{m}{\mu})}{\Gamma(-\frac{m}{\mu})} + \log \left[-\frac{\Gamma(-\mu) B}{\sqrt{2\pi N \phi''(\mu)}} \right] \right) + \mu \frac{\Gamma'(-\mu)}{\Gamma(-\mu)} \right]$$

$$\langle \rho^2 \rangle = -\frac{1}{N} \frac{m}{\mu^2 \phi''(\mu)}$$

which in the case of the REM (61) is fully consistent with (55).

The calculation of the fluctuations of r is easier because in our saddle point calculation the integral over r was performed last. The other fluctuations predicted in (55) are more difficult to recover because the saddle point in μ depends on x which itself depends on r and that both x and r fluctuate. So the fluctuations of μ would combine its own fluctuations with those induced by the fluctuations of x and r . Moreover here there is a single (or may be r) variable μ while in (55) there are p of them. Because of these difficulties we did not calculate the fluctuations of μ . We however believe that, if done correctly, the fluctuations of μ should be consistent with (55).

6. Conclusion

In the present paper we have developed a systematic way of computing finite size effects for the REM. Our approach led to explicit expressions of the leading corrections to the overlaps (39,44) and of the prefactor of the non-integer moments of the partition function (46). We have shown that these results can be interpreted as fluctuations with negative variance (55) of the number and size of the blocks in the broken replica symmetry language.

Our exact expression (62) for the non-integer moments of the partition function can be written in terms of coupled contour integrals over what can be thought of as "complex replica numbers". One step replica symmetry breaking then arises naturally from the saddle point of these integrals without the need of using the conventional replica approach or the Parisi ansatz. We also find that the fluctuations of the "complex replica numbers" in the imaginary direction correspond to the negative variances we observed in the replica calculation.

One can try to extend our approach to calculate the fluctuations and the finite size corrections in a number of other cases such as the REM with complex temperatures [49, 50, 51, 52, 53], generalised random energy models, or directed polymers on a tree [39]. More challenging would be to see whether our approach could give an alternative to the replica method [43, 44, 45] for models of disordered systems [54, 55, 56] such as glasses (see [41] for a recent review) optimisation problems (see [40] for references) which, like the REM, exhibit a one step replicas symmetry breaking (1RSB) [23, 57].

Lastly, generalizing a formula like (62) to some other disordered systems, starting with the Sherrington Kirpatrick model, would certainly improve our understanding of the applicability and limitations of the replica approach.

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Appendices

A. Difference between the REM and the Poisson REM

In this appendix we show that the difference between the original REM and the Poisson REM of section 2 is exponentially small in the system size N .

In the original REM one considers a system with 2^N configurations \mathcal{C} whose energies $E(\mathcal{C})$ are i.i.d. random variables distributed according to a Gaussian distribution

$$P(E) = \frac{1}{\sqrt{\pi N J^2}} \exp \left[-\frac{E^2}{N J^2} \right] \quad (75)$$

and the partition function is

$$Z = \sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})}.$$

The generating function of the partition function is simply given by

$$\langle e^{-tZ} \rangle_{\text{REM}} = \left[\int e^{-te^{-\beta E}} P(E) dE \right]^{2^N} \quad (76)$$

In the Poisson REM of section 2.2, with intensity (10)

$$\rho(E) = 2^N P(E)$$

that we consider in the present paper the same generating function is given by

$$\langle e^{-tZ} \rangle_{\text{Poisson}} = \exp \left[\int (e^{-te^{-\beta E}} - 1) \rho(E) dE \right] \quad (77)$$

In the low temperature phase, to leading order, one can replace $\rho(E)$ by an exponential approximation (32)

$$\rho(E) = 2^N P(E) \simeq A e^{\alpha(E-E_0)}$$

and (77) gives

$$\langle e^{-tZ} \rangle_{\text{Poisson}} = \exp \left[\frac{A e^{-\alpha E_0}}{\beta} \Gamma \left(-\frac{\alpha}{\beta} \right) t^{\frac{\alpha}{\beta}} \right] = \exp \left[-C t^{\frac{\alpha}{\beta}} \right] \quad (78)$$

where

$$C = -\frac{A}{\beta} \Gamma \left(-\frac{\alpha}{\beta} \right) e^{-\alpha E_0}.$$

(Note that $C > 0$ as the approximation (32) is only valid in the low temperature phase i.e. when $\alpha < \beta$ that is when only energies close to the ground state matter). Using the fact that $P(E)$ is normalized one has

$$\log \left[\int e^{-te^{-\beta E}} P(E) dE \right] = \int (e^{-te^{-\beta E}} - 1) P(E) dE - \frac{1}{2} \left[\int (e^{-te^{-\beta E}} - 1) P(E) dE \right]^2 + \dots$$

and one can see that

$$\langle e^{-tZ} \rangle_{\text{REM}} - \langle e^{-tZ} \rangle_{\text{Poisson}} \simeq -\frac{C^2}{2^{N+1}} t^{\frac{2\alpha}{\beta}} \exp \left[-C t^{\frac{\alpha}{\beta}} \right] .$$

Then using the formula (22) one get for $0 < m < \mu = \alpha/\beta$

$$\langle Z^m \rangle_{\text{Poisson}} \simeq C^{\frac{m\beta}{\alpha}} \frac{\beta}{\alpha} \frac{\Gamma(-\frac{\beta m}{\alpha})}{\Gamma(-m)} \quad ; \quad \langle Z^m \rangle_{\text{REM}} - \langle Z^m \rangle_{\text{Poisson}} \simeq -\frac{1}{2^{N+1}} C^{\frac{m\beta}{\alpha}} \frac{\beta}{\alpha} \frac{\Gamma(2 - \frac{\beta m}{\alpha})}{\Gamma(-m)} .$$

We see that the difference has an extra factor 2^{-N} which makes the original REM and the Poisson version coincide to all orders in a $1/N$ expansion.

B. $1/N$ corrections at a saddle point

In this appendix we derive a general formula for the $1/N$ corrections of an arbitrary observable $H(x)$ at a saddle point. Suppose that we want to evaluate, for large N , a ratio of the form

$$\langle H(x) \rangle = \frac{\int dx e^{NF(x)} G(x) H(x)}{\int dx e^{NF(x)} G(x)} \quad (79)$$

by a saddle point method. One has first to locate the saddle point x_c which satisfies

$$F'(x_c) = 0 .$$

Then by expanding F, G, H around x_c one finds

$$\langle H(x) \rangle = H(x_c) + \frac{1}{N} \left[\left(\frac{F'''(x_c)}{2F''(x_c)^2} - \frac{G'(x_c)}{G(x_c) F''(x_c)} \right) H'(x_c) - \frac{1}{2F''(x_c)} H''(x_c) \right] + O\left(\frac{1}{N^2}\right) . \quad (80)$$

One can rewrite this expression as

$$\langle H(x) \rangle = \langle H(x_c + \eta) \rangle_\eta + O\left(\frac{1}{N^2}\right) \quad (81)$$

where

$$\langle \eta \rangle_\eta = \frac{1}{N} \left[\frac{F'''(x_c)}{2F''(x_c)^2} - \frac{G'(x_c)}{G(x_c) F''(x_c)} \right] , \quad (82)$$

$$\langle \eta^2 \rangle_\eta = -\frac{1}{N} \left[\frac{1}{F''(x_c)} \right] . \quad (83)$$

Formulas would remain unchanged if the integration was not along the real axis but along any path in the complex plane. In section 5.5 we in fact use them along contours in the complex plane.

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